§3.1) Normal ordered product

Let V be a super vector space $V = V_{\bar 0} + V_{\bar 1}$

Recall $a_{(n)} = \mathrm{Res}_z\, a(z) z^n$

A formal distribution $a(z) = \sum_{n\in\mathbb{Z}} a_{(n)} z^{-n-1}$ is a field

$a_{(n)} \in \mathrm{End}(V)$

if for any $v \in V$ we have:

$$a_{(n)} v = 0 \quad \text{for } n >> 0$$

Defn:

Normally ordered product of two fields $a(z)$ and $b(z)$ is

$$:a(z)b(z): \; = a(z)_+ b(z) + (-1)^{p(a)p(b)} b(z) a(z)_-$$

where, $a(z)_+ = \sum_{n<0} a_{(n)} z^{-n-1}$ $\quad a(z)_- = \sum_{n\geq 0} a_{(n)} z^{-n-1}$

So the n^{th} coefficient using the expansion: $:a(z)b(z):_{(n)} = \mathrm{Res}_z(:a(z)b(z):z^n)$

$$:a(z)b(z):_{(n)} = \sum_{j=-1}^{-\infty} a_{(j)} b_{(n-j-1)} + (-1)^{p(a)p(b)} \sum_{j=0}^{\infty} b_{(n-j-1)} a_{(j)}$$

Since $a(z)$ & $b(z)$ are fields, the above when applied to v produce only finitely many non-zero terms & thus $:a(z)b(z):$ is a formal distribution. (we can't do this for general formal distri but with two different indeterminates, we can).

Go further & claim:

$:a(z)b(z):$ is even a field!

Given a $v \in V$, let $b_{(s)}v = 0$ for $s \geq M$

$a_{(j)}v = 0$ for $j \geq N$

$b_{(s)}a_{(j)}v = 0$ $(1 \leq j < N)$ for $s \geq K$

for some $M, N, K \in \mathbb{Z}$

Then $:a(z)b(z):_{(n)} v = 0$ for $n \geq M+N+K$

from direct observation

Space of fields with normally ordered prod. form an algebra which is neither commutative (or) associative in general.

- $a(z)$ is a field $\Rightarrow$ $\partial a(z)$ is a field.
- $\partial :a(z)b(z): = :\partial a(z)b(z): + :a(z)\partial b(z):$
  which is a direct calculation from using $(\partial a(z))_{\pm} = \partial(a(z)_{\pm})$
- Recall for $n \in \mathbb{Z}_+$, we had the n^{th}-product of formal dist.

$$a(w)_{(n)}b(w) = \mathrm{Res}_z[a(z), b(w)](z-w)^n.$$

Due to $: :$, we can define n^{th}- product for fields also for $-ve$ n:

$$a(z)_{(-n-1)} b(z) = :\partial^{(n)} a(z) b(z): \qquad n \in \mathbb{Z}_+$$

Defn:

Field in z & w: $\in End(V)$

$$a(z,w) = \sum_{m,n \in \mathbb{Z}} \underbrace{a_{(m,n)}}_{\in End(V)} z^{-m-1} w^{-n-1} \quad \text{s.t.}$$

For each $v \in V$,

$a_{(m,n)} v = 0$ if $m > N$ (resp. $n > N$) for some N indep. of n (resp. m) when $n \ll 0$ (resp. $m \ll 0$)

$a(w,w)$ is a well-defined field.

Another version of Taylor:

Lemma-3.1

For any field $a(z,w)$ & any positive integer N, there exists fields $c^j(w)$ $(0 \leq j \leq N-1)$ and $d^N(z,w)$ s.t.

$$a(z,w) = \sum_{j=0}^{N-1} c^j(w)(z-w)^j + (z-w)^N d^N(z,w)$$

uniquely determined & are $= \partial_z^{(j)} a(z,w)\big|_{z=w}$

Strategy:

Uniqueness of $c^j(w)$: Differentiate the above j times wrt z & let $z=w$

Existence: Do for $N=1$

$$a(z,w) = \underbrace{c^0(w)}_{= a(w,w)} + (z-w)\underbrace{d^0(z,w)}$$

Direct from above formulas

Applying for this gives $N=2$ & higher

$\square$

There were several equiv. prop. but one of them:

$(z-w)^N [a(z), b(w)] = 0$ for $N \gg 0$

Thm 3.1

Let $a(z)$ & $b(z)$ be mutually local, N be a positive integer. Then there exists a field $d^N(z,w)$ s.t. in the region $|z|>|w|$:

$$a(z)b(w) = \sum_{j \geq -N} \frac{a(w)_{(j)} b(w)}{(z-w)^{j+1}} + (z-w)^N d^N(z,w)$$

coeff. of $(z-w)^{-j-1}$ (for $j \geq -N$) is no surprise: Unique

Pf: Use lemma-3.1 to $:a(z)b(w):$ & use
$\partial :a(z)b(z): = :\partial a(z)b(z): + :a(z)\partial b(z):$ &
use

$$\text{OPE} \Rightarrow a(z)b(w) = \sum_{j=0}^{N-1} \frac{a(w)_{(j)}b(w)}{(z-w)^{j+1}} + :a(z)b(w):$$

$$\left(\text{where } a(w)_{(n)}b(w) = \operatorname{Res}_z [a(z),b(w)](z-w)^n\right) \square$$

For OPE of mutually local fields, we can use Taylor upto required order

Turns out, $\exists$ a nice formula for all n^{th}-products of fields $(n \in \mathbb{Z})$:

$$a(w)_{(n)}b(w) =$$
$$\operatorname{Res}_z \left(a(z)\,b(w)\, i_{z,w}(z-w)^n - (-1)^{p(a)p(b)} b(w)a(z)\, i_{w,z}(z-w)^n \right)$$

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§3.2) Dong's lemma

Lemma-3.2

If $a(z), b(z)$ & $c(z)$ are pairwise mutually local fields (resp. formal distri), then $a(z)_{(n)}b(z)$ and $c(z)$ are mutually local fields (resp. formal distri) for all $n \in \mathbb{Z}$

(resp. for $n \in \mathbb{Z}_+$). In particular $:a(z)b(z):$ and $c(z)$ are mutually local provided $a(z), b(z)$ & $c(z)$ are.

<u>Proof:</u> This is by defn. locality of $a(z)_{(n)}b(z)$ & $c(z)$

$$\operatorname{Res}_{z_1}(z-w)^M\Big(i_{z_1,z}(z_1-z)^n a(z_1)b(z)c(w) - (-1)^{p(a)p(b)} i_{z,z_1}(z_1-z)^n b(z)a(z_1)c(w)\Big)$$

$$=\operatorname{Res}_{z_1}(z-w)^M\Big((-1)^{p(c)(p(a)+p(b))}\big(i_{z_1,z}(z_1-z)^n c(w)a(z_1)b(z) - (-1)^{p(a)p(b)} i_{z,z_1}(z_1-z)^n c(w)b(z)a(z_1)\big)\Big)$$

<u>Strategy:</u> (Brute force)

Suffices to show that for $M >> 0$:

$$(z_2-z_3)^M A = (z_2-z_3)^M B \longrightarrow \star$$

where $A = i_{z_1,z_2}(z_1-z_2)^n a(z_1)b(z_2)c(z_3) - (-1)^{p(a)p(b)} i_{z_2,z_1}(z_1-z_2)^n b(z_2)a(z_1)c(z_3)$

$$B = (-1)^{p(c)(p(a)+p(b))}\Big(i_{z_1,z_2}(z_1-z_2)^n c(z_3)a(z_1)b(z_2)$$
$$- (-1)^{p(a)p(b)} i_{z_2,z_1}(z_1-z_2)^n c(z_3)b(z_2)a(z_1)\Big)$$

$\therefore$ Take $z_2 = z$
$z_3 = w$
& Res_z on both sides + use

$a(w)_{(n)} b(w) =$
$$\mathrm{Res}_z \left(a(z)\, b(w)\, i_{z,w}(z-w)^n - (-1)^{p(a)p(b)} b(w)a(z)\, i_{w,z}(z-w)^n \right)$$
to see this is by defn. mutual locality of $a(z)_{(n)} b(z)$ & $c(z)$

Pairwise locality: for $r \gg 0$
$$(z_1-z_2)^r a(z_1)\, b(z_2) = (z_1-z_2)^r (-1)^{p(a)p(b)} b(z_2)\, a(z_1)$$
$$(z_2-z_3)^r b(z_2)\, c(z_3) = (z_2-z_3)^r (-1)^{p(b)p(c)} c(z_3)\, b(z_2)$$
$$(z_1-z_3)^r a(z_1)\, c(z_3) = (z_1-z_3)^r (-1)^{p(a)p(c)} c(z_3)\, a(z_1)$$

Using these pairwise mutually local relations, one can show that the LHS of ⊛ reduces to: (Let $M = 4r$, $n \geq -r$ (I can choose r in this way))

$$\sum_{s=r+1}^{3r} \binom{3r}{s} (z_2 - z_1)^{3r-s} (z_1 - z_3)^s (z_2 - z_3)^r A$$

$|||^{ly}$ RHS will be:

$$\sum_{s=r+1}^{3r} \binom{3r}{s} (z_2 - z_1)^{3r-s} (z_1 - z_3)^s (z_2 - z_3)^r B$$

$\square$

Defn:

- $glf(V)$ = space of all fields (over $\mathbb{C}$) with values in End(V). Under the n^{th}-product, $glf(V)$ is closed $(n \in \mathbb{Z})$. This is called general linear field algebra

- $A \subset glf(V)$ subspace containing identity operator, I_V and closed under n^{th}-products $(n \in \mathbb{Z})$ is called a linear field algebra

- A linear field algebra is called <u>local</u> if it consists of mutually local fields
- A collection of fields <u>generates a field algebra</u> A if A is the minimal field algebra containing the fields we started out with.

Dong's lemma $\Longrightarrow$ A linear field algebra generated by mutually local fields is local.

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§3.3) Wick's thm. and a non-commutative generalization

Normally ordered prod. of more than two fields is defined inductively from right to left.

Eg:- $:a^1(z)a^2(z)a^3(z): \; = \; :a^1(z):a^2(z)a^3(z)::$

So more generally,

$$:a^1(z)a^2(z)\ldots a^N(z): \; = \; :a^1(z):\ldots :a^{N-1}(z)a^N(z)::\ldots:$$

which consists of 2^N terms of the form

$$\pm\, a^{i_1}(z)_+ a^{i_2}(z) \cdots a^{j_1}(z)_- a^{j_2}(z)_- \cdots$$

where $i_1 < i_2 < \dots$, $j_1 > j_2 > \dots$ is a permutation of the set $\{1, 2, \dots, N\}$ and $\pm$ sign is determined by the sign of this permutation but we remove the indices of the even fields first. (super rule)

Wick's thm. is useful for the calculation of OPE of two normally ordered product of "free fields"

Thm:- (Wick)

Let $a^1(z), \dots, a^M(z)$ and $b^1(z), \dots, b^N(z)$ be some collection of fields s.t:

1) $[\underbrace{[a^i(z)_-, b^j(w)]}, c^k(z)] = 0$, $\forall\, i, j, k$ and $c = a$ (or) b.

Notation $\equiv [a^i b^j]$ called "contraction"

2) $[a^i(z)_\pm, b^j(w)_\pm] = 0 \quad \forall\, i, j$

Then one has the following OPE in $|z| > |w|$ region:

$$:a^1(z)\cdots a^M(z): \; :b^1(w)\cdots b^N(w): = \sum_{s=0}^{\mathrm{Min}(M,N)} \sum_{\substack{i_1<\cdots<i_s \\ j_1\neq\cdots\neq j_s}}$$

$$\Big(\pm [a^{i_1} b^{j_1}] \cdots [a^{i_s} b^{j_s}] :a^1(z)\cdots a^M(z) b^1(w)\cdots b^N(w):_{(i_1,\dots,i_s;j_1,\dots,j_s)}\Big)$$

$\pm$: the super rule explained above

$(i_1,\dots,i_s;j_1,\dots,j_s)$ means $a^{i_1}(z),\dots,a^{i_s}(z)$, $b^{j_1}(w),\dots,b^{j_s}(w)$ fields are removed

Strategy:

Quite direct painful calculation where you look at what a typical term on LHS looks like and try to move $a^i(z)_-$ across $b^j(w)_+$ for some i & j & then use commutation relations in the assumptions of the thm, to determine what terms you pick up as you move the terms across $\square$

Defn:

A collection of fields $\{a^\alpha(z)\}$ is called a free field theory if all these fields are mutually local & all the coefficients of the singular parts of the OPE are multiples of the identity

Thus for a free field theory, we can use Wick's thm above to calculate OPE between normally ordered products

Prop. 3.3 (Generalization of Prop. 2.3 but there we had $n \in \mathbb{Z}_+$ & formal distributions)

a) For any two fields $a(w)$ & $b(w)$, $n \in \mathbb{Z}$:

$$(\partial a(w))_{(n)} b(w) = -n a(w)_{(n-1)} b(w)$$

b) For any mutually local fields $a(w)$ & $b(w)$, $n \in \mathbb{Z}$:

$$a(w)_{(n)} b(w) = (-1)^{p(a)p(b)} \sum_{j=0}^{\infty} (-1)^{j+n+1} \partial^{(j)} \left(b(w)_{(n+j)} a(w) \right)$$

c) For any three fields $a(w), b(w)$ & $c(w)$ and for any $m \in \mathbb{Z}_+$, $n \in \mathbb{Z}$ one has:

$$a(w)_{(m)} \left(b(w)_{(n)} c(w) \right) = \sum_{j=0}^{m} \binom{m}{j} \left(a(w)_{(j)} b(w) \right)_{(m+n-j)} c(w)$$
$$+ (-1)^{p(a)p(b)} b(w)_{(n)} \left(a(w)_{(m)} c(w) \right)$$

(No proof)

letting $n=-1$ from above one arrives at the following:

Allows one to calculate OPE of arbitrarily normally ordered prods of pairwise local fields

$$a(z)_{(m)} :b(z)\, c(z): = :\left(a(z)_{(m)} b(z)\right) c(z):$$
$$+ (-1)^{p(a)p(b)} :b(z) \left(a(z)_{(m)} c(z)\right):$$
$$+ \sum_{j=0}^{m-1} \binom{m}{j} \left(a(z)_{(j)} b(z)\right)_{(m-1-j)} c(z)$$

which is the "non-commutative" version of Wick's thm. (The assumptions of Wick's thm need not be assumed but we get some extra correction terms).